

Section 9.4.3 The Direction of $u \times v$

We want to know the direction of $u \times v$ relative to u and v .

Claim: $u \times v$ is $\perp$ to both u and v . Suffices to show that $u \cdot (u \times v) = 0$.

Say $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$

$$\begin{aligned} u \cdot (u \times v) &= u \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= 0. \end{aligned}$$

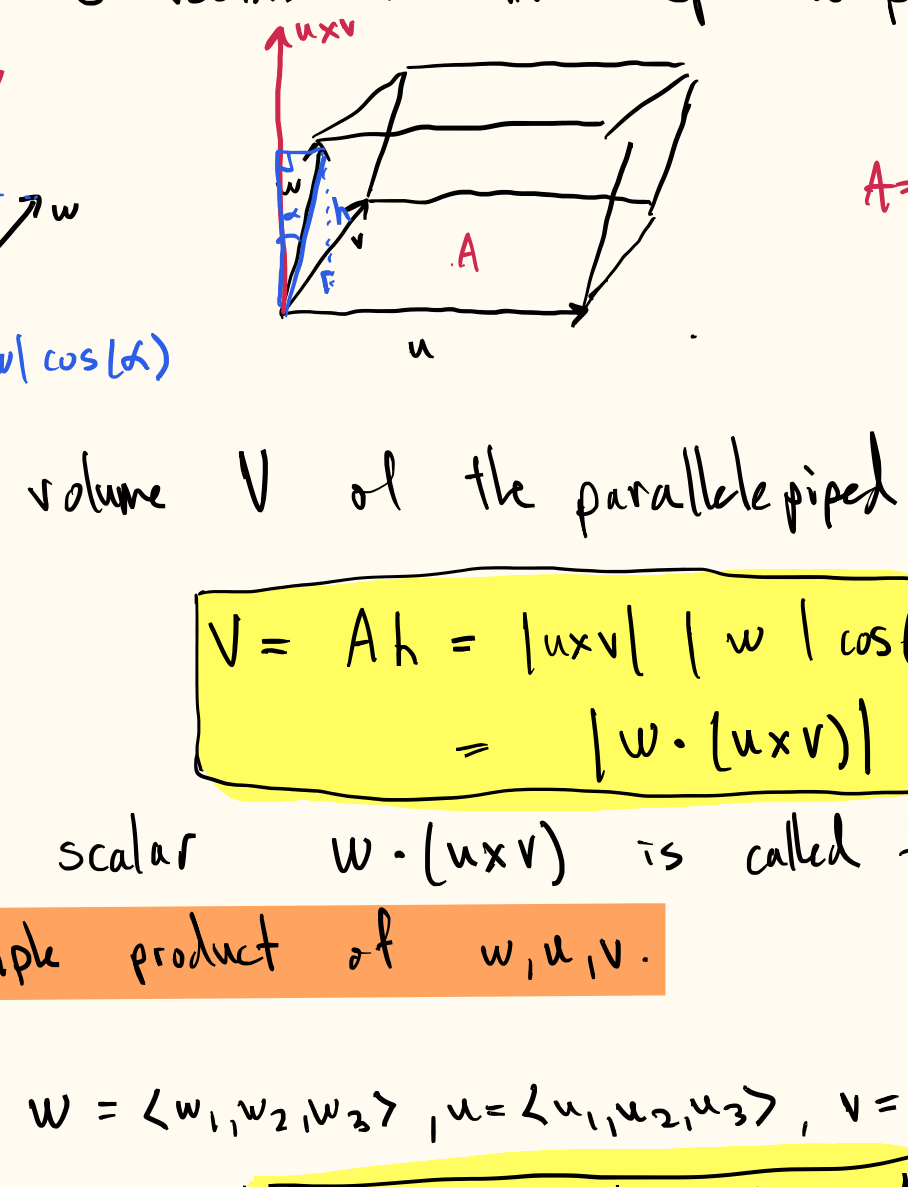
Expand determinant and use properties of dot product.

The determinant of a matrix with 2 of the same row or column is always 0.

Similarly,

$$\begin{aligned} v \cdot (u \times v) &= v \cdot (-v \times u) \\ &= (-1)(v \cdot (v \times u)) \\ &= 0. \end{aligned}$$

It is also possible to show that $(u, v, u \times v)$ satisfies the right-hand rule



Geometric Definition of $u \times v$ Let $u, v \in \mathbb{R}^3$. Then $u \times v$ is the unique vector satisfying:

- $u \times v$ is perpendicular to both u and v and $(u, v, u \times v)$ satisfies the right hand rule.
- $|u \times v| = |u||v|\sin\theta$
= area of parallelogram spanned by u & v where $0 \leq \theta \leq \pi$ is the angle between u & v .

Scalar Triple Product

Any 3 vectors in $\mathbb{R}^3$ span a parallelepiped

$$h = |w| \cos(\alpha)$$

$$A = |u \times v|$$

The volume V of the parallelepiped is:

$$\begin{aligned} V &= Ah = |u \times v| (|w| \cos(\alpha)) \\ &= |w \cdot (u \times v)| \end{aligned}$$

The scalar $w \cdot (u \times v)$ is called the **scalar triple product** of w, u, v .

If $w = \langle w_1, w_2, w_3 \rangle$, $u = \langle u_1, u_2, u_3 \rangle$, $v = \langle v_1, v_2, v_3 \rangle$, then

$$w \cdot (u \times v) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Activity 9.4.4

- Complete Activity 9.4.4 and discuss w/ your group
- Class discussion.

(a)

$$\begin{aligned} u \times v &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & -1 \\ -2 & -2 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 5 & -1 \\ -2 & 1 \end{vmatrix} \hat{i} - \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} \hat{j} + \begin{vmatrix} 3 & 5 \\ -2 & -2 \end{vmatrix} \hat{k} \\ &= \langle 3, -5, -16 \rangle \end{aligned}$$

$\frac{u \times v}{|u \times v|}$ and $\frac{v \times u}{|u \times v|}$ are two unit vectors $\perp$ to both u and v .

(b)

$$\text{Volume} = |w \cdot (u \times v)|$$

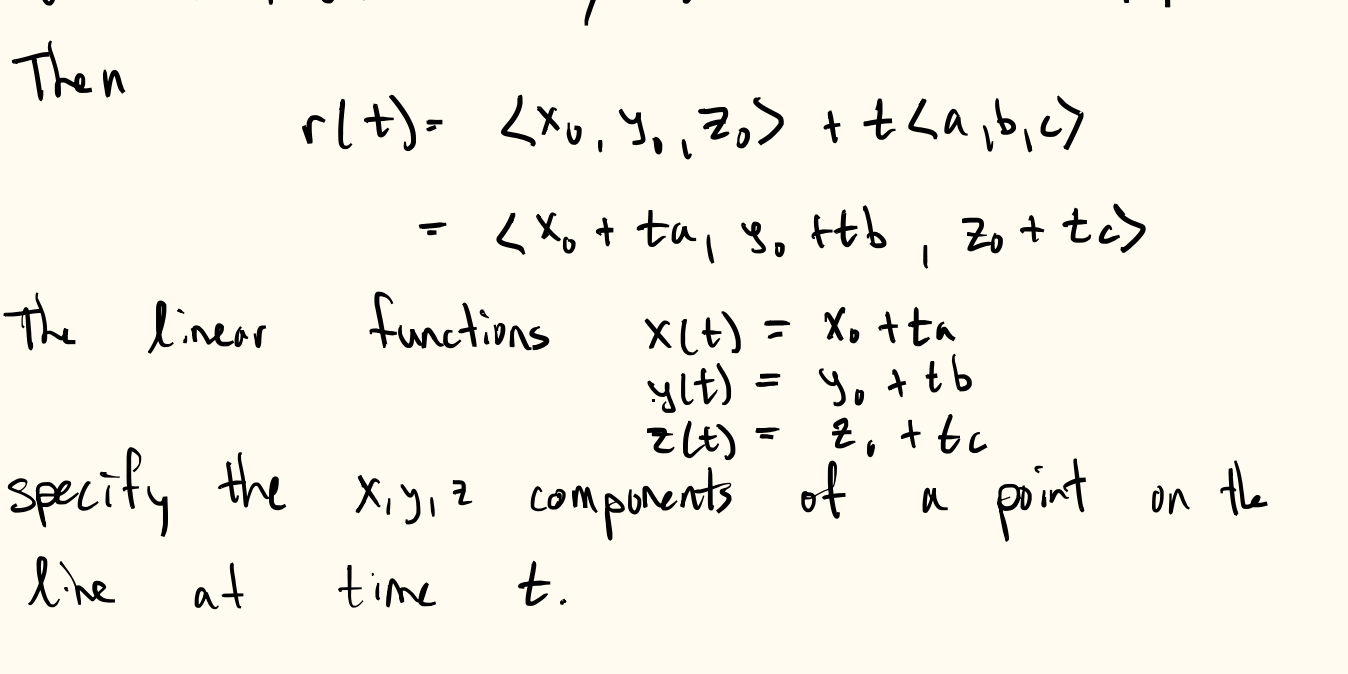
$$= \begin{vmatrix} 3 & 3 & 1 \\ 3 & 5 & -1 \\ 2 & -2 & 1 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 5 & -1 \\ -2 & 1 \end{vmatrix} - 3 \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 3 & 5 \\ -2 & -2 \end{vmatrix}$$

$$= |3 \cdot 3 - 3 \cdot 5 + (-16)|$$

$$= 22.$$

(c) Draw a picture:



From the picture, one possible normal vector is $n = \vec{QP} \times \vec{QR}$.

Compute:

$$\vec{QP} \times \vec{QR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 0 \\ 6 & -1 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 \\ -1 & -2 \end{vmatrix} \hat{i} - \begin{vmatrix} 2 & 0 \\ 6 & -2 \end{vmatrix} \hat{j} + \begin{vmatrix} 2 & -1 \\ 6 & -1 \end{vmatrix} \hat{k}$$

$$= \langle 2, 4, 4 \rangle.$$

(d)

Use the right-hand rule.

(e)

The vectors lie in the same plane if and only if the volume of the parallelepiped is zero.

$$\text{Compute } c \cdot (a \times b) = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 3 & 2 \\ 2 & -1 & 4 \end{vmatrix} = -1 \cdot 2 = 0.$$

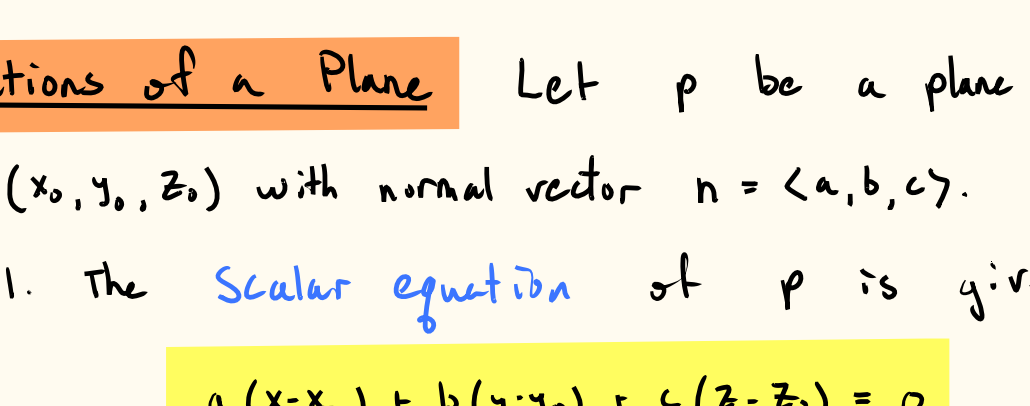
the vectors are coplanar.

End of Section 9.4.

Section 9.5.1 Lines and Planes

Definition 9.5.3

A **line** through a point P in the direction of a vector v is the set of terminal points of all vectors parallel to v with initial point P .



Vector form of a Line

The **vector form** of the line through P in the direction of v is the vector-valued function

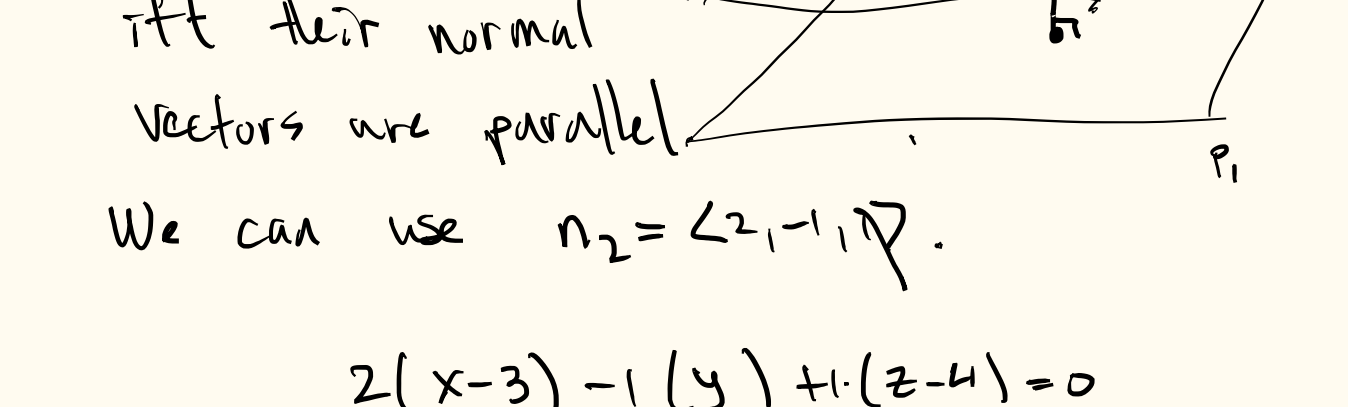
$$r(t) = \vec{OP} + tv.$$

Reading Debrief

- Discuss Activity 9.5.2 w/ the person next to you.
- Questions about anything from Section 9.5.1?

Activity 9.5.2. Let $P_1 = (1, 2, -1)$ and $P_2 = (-2, 1, -2)$. Let L be the line in $\mathbb{R}^3$ through P_1 and P_2 , which is the same line as in Activity 9.5.1.

- Find a vector equation of L in the form $r(t) = r_0 + tv$.
- Does the point $(1, 2, 1)$ lie on L ? If so, what value of t results in this point?
- Consider the vector equation $s(t) = (-5, 0, 1) + t(4, 2, 2)$. What is the direction of the line given by $s(t)$? Is this line parallel to line L ?
- Do $r(t)$ and $s(t)$ represent the same line, L ? Explain.



(d) Solve

$$\langle 1, 2, -1 \rangle = s(t) = \langle -5, 0, -3 \rangle + t \langle 4, 2, 2 \rangle$$

Since $s(1) = \langle 1, 2, -1 \rangle$, $(1, 2, -1)$ lies on $s(t)$. Since s and r are parallel and intersect, they are the same.

Section 9.5.2 Parametric Equations of a Line

Suppose $r(t) = r_0 + tv$ is a line thru $r_0 = \langle x_0, y_0, z_0 \rangle$ w/ direction $v = \langle a, b, c \rangle$

Then

$$\begin{aligned} r(t) &= \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle \\ &= \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle \end{aligned}$$

The linear functions

$$\begin{aligned} x(t) &= x_0 + ta \\ y(t) &= y_0 + tb \\ z(t) &= z_0 + tc \end{aligned}$$

specify the x, y, z components of a point on the line at time t .

Parametric Eq of a Line

The functions

$$x(t) = x_0 + at, \quad y(t) = y_0 + bt, \quad z(t) = z_0 + ct$$

are called the **parametric eqs of a line**.

Activity 9.5.3

- Complete 9.5.3 and discuss w/ the person next to you.
- Class discussion.

Activity 9.5.3. Let $P_1 = (1, 2, -1)$ and $P_2 = (-2, 1, -2)$, and let L be the line in $\mathbb{R}^3$ through P_1 and P_2 .

- Find parametric equations of the line L .
- Does the point $(2, 0, 2)$ lie on L ? If so, what value of t results in this point?
- Consider another line, K , whose parametric equations are $x(t) = 11 + 4t$, $y(t) = 1 - 3t$, $z(t) = 3 + 2t$.
- What is the direction of the line K ?
- Do the lines L and K intersect? If so, provide the point of intersection and the t and s values, respectively, that result in the point. If not, explain why.

(a) By Activity 9.5.2, r and s are different parametrizations of the same line.

Using $s(t) = \langle -5 + 4t, 0 - 3t, -3 + 2t \rangle$, one possible set of parametric equations is

$$\begin{aligned} \textcircled{1} \quad x(t) &= -5 + 4t \\ \textcircled{2} \quad y(t) &= -3t \\ \textcircled{3} \quad z(t) &= -3 + 2t \end{aligned}$$

(b) No. If $(1, 2, 1)$ lies on L , then eq. $\textcircled{2}$ implies that $t = 1$. But $z(1) = -3 + 2(1) = -1$.

(c) The direction is $\langle 4, -3, 2 \rangle$.

(d) If they do, then there is a solution to the system of equations

$$\begin{cases} -5 + 4t = 11 + 4s \\ -3t = 1 - 3s \\ -3 + 2t = 3 + 2s \end{cases} \quad \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

solve by substitution or linear algebra.

Linear algebraic method for solving system:

$$\begin{aligned} R_1 \quad 4t - 4s &= 16 \\ R_2 \quad -3t + 3s &= 1 \\ R_3 \quad 2t - 2s &= 6 \end{aligned} \quad \text{Row-reduction algorithm:}$$

$$\begin{aligned} R_1 \quad & \begin{bmatrix} 4 & -4 & 16 \\ -3 & 3 & 1 \\ 2 & -2 & 6 \end{bmatrix} \xrightarrow{R_2 + R_1 \rightarrow R_2} \begin{bmatrix} 4 & -4 & 16 \\ -2 & 3 & 1 \\ 2 & -2 & 6 \end{bmatrix} \xrightarrow{R_3 - 2R_1 \rightarrow R_3} \begin{bmatrix} 4 & -4 & 16 \\ -2 & 3 & 1 \\ 0 & 1 & 7 \end{bmatrix} \\ R_2 \times 2 \rightarrow R_2 & \begin{bmatrix} 4 & -4 & 16 \\ -4 & 6 & 2 \\ 0 & 1 & 7 \end{bmatrix} \xrightarrow{R_1 + R_2 \rightarrow R_1} \begin{bmatrix} 0 & 2 & 18 \\ -4 & 6 & 2 \\ 0 & 1 & 7 \end{bmatrix} \\ -2R_2 \rightarrow R_2 & \begin{bmatrix} 0 & 2 & 18 \\ 2 & -3 & -1 \\ 0 & 1 & 7 \end{bmatrix} \xrightarrow{R_1 - 2R_2 \rightarrow R_1} \begin{bmatrix} 0 & 2 & 18 \\ 2 & -3 & -1 \\ 0 & 1 & 7 \end{bmatrix} \\ R_1 \times \frac{1}{2} \rightarrow R_1 & \begin{bmatrix} 0 & 1 & 9 \\ 2 & -3 & -1 \\ 0 & 1 & 7 \end{bmatrix} \xrightarrow{R_2 - R_1 \rightarrow R_2} \begin{bmatrix} 0 & 1 & 9 \\ 2 & -4 & -10 \\ 0 & 1 & 7 \end{bmatrix} \\ R_2 \times \frac{1}{2} \rightarrow R_2 & \begin{bmatrix} 0 & 1 & 9 \\ 1 & -2 & -5 \\ 0 & 1 & 7 \end{bmatrix} \xrightarrow{R_2 - R_1 \rightarrow R_2} \begin{bmatrix} 0 & 1 & 9 \\ 1 & -3 & -14 \\ 0 & 1 & 7 \end{bmatrix} \\ R_2 \times (-1) \rightarrow R_2 & \begin{bmatrix} 0 & 1 & 9 \\ -1 & 3 & 14 \\ 0 & 1 & 7 \end{bmatrix} \xrightarrow{R_2 + R_1 \rightarrow R_2} \begin{bmatrix} 0 & 1 & 9 \\ 0 & 4 & 23 \\ 0 & 1 & 7 \end{bmatrix} \\ R_2 \times \frac{1}{4} \rightarrow R_2 & \begin{bmatrix} 0 & 1 & 9 \\ 0 & 1 & \frac{23}{4} \\ 0 & 1 & 7 \end{bmatrix} \xrightarrow{R_2 - R_1 \rightarrow R_2} \begin{bmatrix} 0 & 1 & 9 \\ 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 7 \end{bmatrix} \end{aligned}$$

This says $0 = -16$. Nonsense!

By using substitution or linear algebra, there is no solution and the lines do not intersect.

Section 9.5.3 Planes in Space

Definition 9.5.6

A **plane** p through a point P_0 with normal vector n is the set of terminal points of all vectors with initial point P_0 that are perpendicular to n .

How can we determine when an arbitrary point $P = (x, y, z)$ lies on p ?

Notice: P lies in $p \iff n$ is perp. to $\vec{P_0P}$

$$\iff n \cdot \vec{P_0P} = 0$$

$$\iff \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\iff a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Equations of a Plane

Let p be a plane through $P_0 = (x_0, y_0, z_0)$ with normal vector $n = \langle a, b, c \rangle$.

- The **scalar equation** of p is given by

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

or

$$ax + by + cz = d \quad \text{where } d = ax_0 + by_0 + cz_0$$

- Let $P = (x, y, z)$ be an arbitrary point. The **vector equation** of p is given by

$$n \cdot \vec{P_0P} = 0.$$

Ex

$$12x + 15y + 4z = 5$$

normal vector $n = \langle 12, 15, 4 \rangle$

A point on the plane: $(0, 0, 5/4)$

Activity 9.5.4

- Complete 9.5.4 and discuss w/ the person next to you.
- Class discussion.

Activity 9.5.4. Write a scalar equation of the plane p_1 passing through the point $(0, 2, 4)$ and perpendicular to the vector $n = \langle 2, -1, 1 \rangle$.

- Write a scalar equation of the plane p_2 that is parallel to p_1 and passing through the point $(3, 0, 5)$. (Hint: Consider normal vectors of the planes.)
- Write a parametric description of the line l passing through the point $(2, 0, 2)$ and perpendicular to the plane p_1 described by the equation $x - 2y - 2z = 7$.
- Find the value of t at which l intersects the plane p_1 .

(a) $P_0 = (0, 2, 4)$ $n = \langle 2, -1, 1 \rangle$

$$2(x - 0) - 1(y - 2) + 1(z - 4) = 0$$

$$2x - y + z = 2$$

(b)

Plug in $(2, 0, 2)$. $6 = 2$

Doesn't lie on the plane. $n_2 = ?$

(c)

Draw a picture!

Two planes are parallel iff their normal vectors are parallel.

We can use $n_2 = \langle 2, -1, 1 \rangle$.

$$2(x - 3) - 1(y) + (z - 4) = 0$$

$$2x - y + z = 10$$

(d)

the direction of l

is $v = \langle 1, 2, -2 \rangle$

(normal vector for p_1)

Answer: $r(t) = \langle 2, 0, 2 \rangle + t \langle 1, 2, -2 \rangle$

(e)

Plug $x(t), y(t), z(t)$ into the plane eq.

$$(2 + t) + 2(2t) - 2(2 - 2t) = 7$$

So $t = 1$. Point of intersection

$$r(1) = \langle 3, 2, 0 \rangle$$

Activity 9.5.5

- Complete 9.5.4 and discuss w/ the person next to you.
- Class discussion.

Activity 9.5.5. Let $P_1 = (1, 2, -1)$, $P_2 = (-2, 1, -2)$, and $P_3 = (0, 1, 3)$ and let p be the plane containing P_1, P_2 , and P_3 .

- Determine the components of the vectors $\vec{P_1P_2}$ and $\vec{P_1P_3}$.
- Find a scalar equation of the plane p .
- Consider a vector u with scalar equation $-3x - (1 + 4y + 3) + 2(z - 5) = 0$. Find two different points on plane p and write a vector w in the plane p .
- What is the angle between the planes p and q ?

(a)

$$u = \vec{P_1P_2} = P_2 - P_1 = \langle -1, -1, 4 \rangle$$

$$v = \vec{P_1P_3} = P_3 - P_1 = \langle -1, -1, 4 \rangle$$

(b)

$$n = u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 4 \\ -1 & -1 & 4 \end{vmatrix} = 2 \begin{vmatrix} \hat{i} & \hat{j} \\ -1 & -1 \end{vmatrix}$$

$$= 8\hat{i} + 2\hat{j}$$

(c)

$$8(x - 1) + 2(z + 1) = 0$$

$$\text{or } 8x + 2z = 6$$

(d)